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AN APPLICATION OF THE FREE CONVOLUTION

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0. Introduction

In [Vo1], Voiculescu began studying the operator algebra free products from the probabilistic point of view. His idea is to look at free products as an analogue of tensor products and to develop a corresponding highly noncommutative probabilistic framework, where freeness is given as the notion of independence (see the monograph [VDN]). It has been introduced in [Vo2] the operation of the additive free convolution as analogue of the usual convolution. In order to compute it, it was also introduced the R-transform which linearlizes the additive free convolution. The definition of the R-transform goes in terms of a certain family of formal Toeplitz operators, which is given in [Vo2] by Voiculescu. An alternative, combinatorial approach to the R-transform was found by Speicher in [Sp]. The most important advantages of this combinatorial approach is that it can be generalized in a straightforward way to multi-dimensional situations as in [Ni]. And they have been developed much more the conbinatorial approaches to free random variables. The machinery of the R-transform was found independently and simultaneously by Woess in [Wo], by Soardi in [So], and by Cartwight and Soardi in [CS1], [CS2], from the studies of the random walks on free product groups to obtain the walk generating function or the Plancherel measures.

The spectral theory of the infinite graphs such as the homogenuous tree or the infinite distance regular graphs, has been studied in [BMS], [FP], [IP], and [KS], for example. The survey on the spectra of infinite graphs is now available in [NW]. Especially, many authors have contributed to spectral theory and harmonic analysis for the homogeneous tree T_m. The results concerning with its harmonic analysis are subsumed in [FTP]. If m is even, then T_m is the Cayley graph of a free group, and many papers have dealed with this structure. The ancestor is Kesten [Ke], who calculates the closed walk generating function of the transition operator. In [Vo4], Voiculescu has also treated it by using the R-transform, which is called generally free harmonic analysis in [VDN].

In this lecture, we make a breif introduction of the free probability theory and show some exapmles of free harmonic analysis on a free family of projections. We treat the typical two cases of them. The first one is $\{p_i\}_{i=1,2,\dots,n}$ with the same state $\phi(p_i) = \alpha$, and we consider the operator $\lambda \sum_{i=1}^{n} p_i$. The other is $\{p, q\}$ with different states $\phi(p) = \alpha$ and $\phi(q) = \beta$, and we consider the operator $\lambda p + \mu q$. The former corresponds to the radial case in the theory of representations (see, for example, [FTP] or [Co]) and the latter to the semiradial case as in [CT1].

1. Noncommutative probability spaces

This section contains preliminaries concerning with noncommutative probability spaces and free random variables. Recall that a usual probability space is (Ω, Σ, ν), where Ω is a base space, Σ is a σ-algebra and ν is a probability measure (i.e. positive and satisfying $\nu(\Omega) = 1$). A random variable is a mesurable function $f : \Omega \to \mathbb{C}$, and if f is integrable then its expectation $E(f)$ is given by

$$E(f) = \int_{\omega \in \Omega} f d\nu(\omega). \tag{1.1}$$

We can consider a noncommutative probability space in a purely algebraic frame as an analogue of the above usual probability space.

Definition 1.1. A *noncommutative probability space* is $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\phi(1) = 1$. We say that $(\mathcal{A}, \phi)$ is a *C^*-probability space* when, in addition, $\mathcal{A}$ is a C^*-algebra and ϕ is a state.

One can define independence in a noncommutative probability space as generalization of the usual definition, which is based on the tensor product of algebras. Instead of the tensor product the reduced free product, we can introduce a much more noncommutative independence called free, which is due to Voiculescu and explained below.

Definition 1.2. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $\mathcal{A}_i$ be subalgebra of $\mathcal{A}$ containing the identity element of $\mathcal{A}$, $1 \in \mathcal{A}_i \subset \mathcal{A}$, for $i \in I$. We say that the family $(\mathcal{A}_i)_{i \in I}$ is *free* if

$$\phi(x_1 x_2 \cdots x_n) = 0 \tag{1.2}$$

whenever $x_j \in \mathcal{A}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(x_j) = 0$ for all j.

A family of subsets $X_i \subset \mathcal{A}$ (resp. elements $x_i \in \mathcal{A}$) will be called *free* if the family of subalgebras $\mathcal{A}_i$ generated by $\{1\} \cup X_i$ (resp. $\{1, x_i\}$) is free.

Example 1.3. For a discrete group G, consider the left regular representation of G on $\ell^2(G)$, given by $g \mapsto \lambda_G(g)$, where $(\lambda_G(g)\xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$ and $\xi \in \ell^2(G)$. The reduced group C^*-algebra of G is

$$C_r^*(G) = \overline{\text{span}}^{\|\ \|}\{\lambda_G(g) : g \in G\}, \tag{1.3}$$

the operator norm closed $*$-algebra generated by $\{\lambda_G(g) : g \in G\}$. It has the canonical faithful tracial state $\tau_G(\cdot) = \langle \cdot\ \delta_e | \delta_e \rangle$ where δ_e is the characteristic function of the identity e of G.

If G is the free product of a family $\{G_j\}_{j=1,2,\cdots,k}$ of discrete groups and

$$A_j = \overline{\text{span}}^{\|\ \|}\{\lambda_G(g) : g \in G_j\} \tag{1.4}$$

then (A_j) is free in the C^*-probability space $(C_r^*(G), \tau_G)$.

Definition 1.4. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A *random variable* is an element $x \in \mathcal{A}$. The *distribution* of x is the linear functional ν_x on $\mathbb{C}[X]$ (the algebra of complex polynomials in the variable X), defined by

$$\nu_x(P(X)) = \phi(P(x)), \quad \text{for all } P \in \mathbb{C}[X]. \tag{1.5}$$

Note that the distribution of a random variable $x \in \mathcal{A}$ is nothing more than a way of discribing the simple moments.

Remark 1.5. In a C^*-probability space $(\mathcal{A}, \phi)$, if x is a self-adjoint element of $\mathcal{A}$ then the distribution of x, ν_x, extends to a compactly supported measure on $\mathbb{R}$, namely there exists a unique probability measure $d\nu_x$ on $\mathbb{R}$ such that

$$\int P(t)d\nu_x(t) = \phi(P(x)). \tag{1.6}$$

Thus, for a self-ajoint element x, we simply call ν_x the probability measure of x in this paper.

Definition 1.6. If x_1 and x_2 are free random variables with distributions ν_{x_1} and ν_{x_2} then the distribution $\nu_{x_1+x_2}$ (which depends only on ν_{x_1} and ν_{x_2}) is called the *additive free convolution* of ν_{x_1} and ν_{x_2}, and denoted by $\nu_{x_1} \boxplus \nu_{x_2}$.

As a tool for computing the additive free convolution, Voiculescu introduced the notion of the R-transform [Vo2] instead of the Fourier transform of a distrubution in the usual probability theory.

Definition 1.7. For a distribution ν on $\mathbb{C}[X]$, we consider the formal power series of ζ^{-1}

$$G_\nu(\zeta) = \zeta^{-1} + \sum_{k=1}^{\infty} \nu(X^k)\zeta^{-k-1} \tag{1.7}$$

and the formal power series of z

$$K_\nu(z) = z^{-1} + \sum_{k=0}^{\infty} \alpha_{k+1} z^k, \tag{1.8}$$

where $G_\nu(\zeta)$, $K_\nu(z)$ are mutually inverse, that is,

$$G_\nu(K_\nu(z)) = z \quad \text{and} \quad K_\nu(G_\nu(\zeta)) = \zeta \tag{1.9}$$

The *R-transform* $R_\nu(z)$ of the distribution ν is defined by the following form:

$$R_\nu(z) = K_\nu(z) - \frac{1}{z} = \sum_{k=0}^{\infty} \alpha_{k+1} z^k. \tag{1.10}$$

Example 1.8. Suppose $p \in \mathcal{A}$ is a projetion in a C^*-probability space $(\mathcal{A}, \phi)$ with $\phi(p) = \alpha$. Let ν be the distribution of p. Then we have, for $k \geq 1$,

$$\nu(X^k) = \phi(p^k) = \phi(p) = \alpha \tag{1.11}$$

and

$$G_\nu(\zeta) = \zeta^{-1} + \alpha \sum_{k=1}^{\infty} \zeta^{-k-1} = \frac{\zeta - (1-\alpha)}{\zeta(\zeta - 1)}. \tag{1.12}$$

Thus we can find the R-transform of ν by

$$z = G_\nu(R_\nu(z)) = \frac{R_\nu(z) - (1-\alpha)}{R_\nu(z)(R_\nu(z) - 1)} \tag{1.13}$$

and obtain

$$R_p(z) = \frac{1}{2z}\left\{(z-1) + \sqrt{(z-1)^2 + 4\alpha z}\right\} \tag{1.14}$$

where the analytic square root is choosen as $\lim_{z\to 0} R_p(z) = \alpha$.

Theorem 1.9. *For free random variables x_1, x_2, we have*

$$R_{\nu_{x_1} \boxplus \nu_{x_2}}(z) = R_{\nu_{x_1}}(z) + R_{\nu_{x_2}}(z). \tag{1.15}$$

This theorem says that the R-transform linearizes the additive free convolution. Hence we can regrad it as a free analogue of the logarithm of the Fourier transform, or of the cumulants generating series, in the ususal probability theory.

Futhermore the following formulae can be easily shown:

$$R_{\nu_{\gamma x}}(z) = \gamma R_{\nu_x}(\gamma z), \tag{1.16}$$

$$R_{\nu_{x+\gamma \cdot 1}}(z) = R_{\nu_x}(z) + \gamma \quad \text{for} \quad \gamma \in \mathbb{C}. \tag{1.17}$$

Remark 1.10. If ν is the distribution of an element, $x \in A$, of a C^*-probability space $(\mathcal{A}, \phi)$, then for $|\zeta| > ||x||$,

$$G_{\nu_x}(\zeta) = \phi\left((\zeta I - x)^{-1}\right). \tag{1.18}$$

If, in addition, x is a self-adjoint element, then ν_x is a probability measure compactly supported on $\sigma(x) \subset \mathbb{R}$, the spectrum of x, and thus

$$G_{\nu_x}(\zeta) = \int_{\sigma(a)} \frac{d\nu_x(t)}{\zeta - t} \tag{1.19}$$

is precisely the Cauchy transform of ν, which is an analytic function defined for ζ in a neighborhood of ∞. Hence, in such a case, we can recover the measure ν_x from G_{ν_x} by using the Stieltjes inversion formula (see [Ak]).

2. The linear combinations of a free family of projetions

Let p be a projections in a C^*-probability space $(\mathcal{A}, \phi)$ with $\phi(p) = \alpha$. Then the R-transform of the projection p can be given by Example 1.8. By the property of the R-transform for a dilation, we have

$$R_{\lambda p}(z) = \frac{1}{2z}\left\{(\lambda z - 1) + \sqrt{(\lambda z - 1)^2 + 4\alpha\lambda z}\right\}. \tag{2.1}$$

Let $\{p_i\}_{i=1,2,\cdots,n}$ be a free family of projections with $\phi(p_i) = \alpha_i$ for each i. We consider the linear combination,

$$\ell = \sum_{i=1}^{n} \lambda_i p_i \tag{2.2}$$

of these projections, where λ_i is assumed to be positive. The R-transform of the element ℓ is now given by

$$R_\ell(z) = \sum_{i=1}^{n} \frac{1}{2z} \left\{ (\lambda_i z - 1) + \sqrt{(\lambda_i z - 1)^2 + 4\alpha_i \lambda_i z} \right\}. \tag{2.3}$$

Hence we have

$$\begin{aligned} K_\ell(z) &= R_\ell(z) + \frac{1}{z} \\ &= -\left(\frac{n-2}{2z}\right) + \frac{1}{2z} \sum_{i=1}^{n} \left\{ \lambda_i z + \sqrt{(\lambda_i z - 1)^2 + 4\alpha_i \lambda_i z} \right\}. \end{aligned} \tag{2.4}$$

In order to obtain $G_\ell(\zeta)$, it will be required to invert the function K_ℓ, that is to solve the equation $\zeta = K_\ell(z)$ in z.

It is immediately seen that $G_\ell(\zeta)$ is an algebraic function, but in general case, it can not be solved in radicals. However, this can be done, for instance, in the cases where at most two different square roots will appear in the right hand side of the equation (2.4). That is, in the cases where the family $\{(\alpha_i, \lambda_i)\}_{i=1,2,\cdots,n}$ is constituted from at most two different pairs. From now on, we shall concentrate our attention upon the following typical two cases and find the probability measure of the random variable ℓ in each case:

Case 1) $(\alpha_i, \lambda_i) = (\alpha, \lambda)$ for $i = 1, 2, ..., n$, with $n \geq 2,\ 0 < \alpha < 1,\ \lambda > 0$,
Case 2) $n = 2$ and $\{(\alpha_i, \lambda_i)\}_{i=1,2}$ with $0 < \alpha_i < 1,\ \lambda_i > 0$.

First we shall investigate Case 1). In this case, the equation $\zeta = K_\ell(z)$ becomes

$$\zeta = -\left(\frac{n-2}{2z}\right) + \frac{n}{2z}\left(\lambda z + \sqrt{(\lambda z - 1)^2 + 4\alpha\lambda z}\right), \tag{2.5}$$

which yields the quadratic equation in z that

$$\zeta(\zeta - n\lambda)z^2 + ((n-2)\zeta + n\lambda(1 - n\alpha))\, z + (1 - n) = 0. \tag{2.6}$$

We put

$$\gamma_\pm = \lambda\{(n-2)\alpha + 1\} \pm 2\lambda\sqrt{(n-1)\alpha(1-\alpha)} \tag{2.7}$$

then the G-series of the element ℓ can be obtained as

$$G_\ell(\zeta) = \frac{-\{(n-2)\zeta + n\lambda(1 - n\alpha)\} + n\sqrt{(\zeta - \gamma_+)(\zeta - \gamma_-)}}{2\zeta(\zeta - n\lambda)}, \tag{2.8}$$

where the branch of the analytic square root should be determined by

$$\mathrm{Im}\zeta > 0 \quad \Longrightarrow \quad \mathrm{Im}G(\zeta) \leq 0. \tag{2.9}$$

Here we note that we can have the inequalities

$$0 \leq \gamma_- < \gamma_+ \leq n\lambda. \tag{2.10}$$

We shall calculate the probability measure ν of the self-adjoint element ℓ by using the Stieltjes inversion formula on G_ℓ. For this purpose, we should take into accounts the some algebraic structure of $G_\ell(\zeta)$. $\zeta = 0$ is removable singularity if $1 - n\alpha \leq 0$. When $1 - n\alpha > 0$, it is a simple pole with residue $1 - n\alpha$. Similary, $\zeta = n\lambda$ is removable singularity if $1 - n(1-\alpha) \leq 0$. When $1 - n(1-\alpha) > 0$, it is a simple pole with residue $1 - n(1-\alpha)$. The Stieltjes inversion formula says that ν is absolutely continuous with respect to Lebesgue measure where $G_\ell(\zeta)$ has non-zero imaginary part on the real axis, in our case, on the interval $[\gamma_-, \gamma_+]$. The density of the absolutely continuous part of the measure ν with respect to Lebesgue measure can be given by

$$f(t) = -\frac{1}{\pi} \lim_{\varepsilon \to +0} \mathrm{Im}\ G_\ell(t + i\varepsilon) = \frac{-n\sqrt{-(t-\gamma_+)(t-\gamma_-)}}{2\pi t(t - n\lambda)}. \tag{2.11}$$

for $t \in [\gamma_-, \gamma_+]$. From the above observations, we have the following theorem:

Theorem 2.1. *Let $\{p_i\}_{i=1,2,\cdots,n}$ be a free family of projections with $\phi(p_i) = \alpha$ for all i and we put $\ell = \lambda \sum_{i=1}^n p_i$ where $\lambda > 0$. Then the distribution ν for the element ℓ is given by*

$$\begin{aligned} d\nu = {} & \frac{-n\sqrt{-(t-\gamma_+)(t-\gamma_-)}}{2\pi t(t - n\lambda)} \chi_{[\gamma_-,\gamma_+]} dt \\ & + \max(0, 1 - n\alpha)\delta_0 + \max(0, 1 - n(1-\alpha))\delta_{n\lambda}, \end{aligned} \tag{2.12}$$

where dt denotes the Lebesgue measure, δ_t is the Dirac unit mass at t, and χ_I means the characteristic function for the interval I.

Next we shall consider Case 2). That is $\ell = \lambda p + \mu q$ where p and q are free projections with $\phi(p) = \alpha$ and $\phi(q) = \beta$, and let λ and μ are positive scalars. The structure of the C^*-algebra generated by the projections p, q and the identity,

$C^*(p,q,1)$ has been investigated in [ABH] and a certain spectral analysis is also studied in [Vo4] and [VDN]. We shall, however, give the measures directly by using the R-transform here.

In this case, the equation $\zeta = K_\ell(z)$ becomes

$$\zeta = \frac{1}{2z}\left\{(\lambda+\mu)z + \sqrt{(\lambda z-1)^2+4\alpha\lambda z} + \sqrt{(\mu z-1)^2+4\beta\mu z}\right\}. \tag{2.13}$$

This equation yeilds the following equation:

$$\begin{aligned}&\left(\{(\lambda+\mu)-2\zeta\}^2 z^2 - \{(\lambda z-1)^2+4\lambda\alpha z\} - \{(\mu z-1)^2+4\mu\beta z\}\right)^2\\ =&4\left\{(\lambda z-1)^2+4\lambda\alpha z\right\}\left\{(\mu z-1)^2+4\mu\beta z\right\},\end{aligned} \tag{2.14}$$

which is rather large but its degree in z might be at most 4. After some more tedious calculation, we can see that it will be reduced to the quadratic equation $Az^2 + Bz + C = 0$, where

$$\begin{aligned}A &= \zeta(\zeta-\lambda)(\zeta-\mu)(\zeta-\lambda-\mu),\\ B &= \{\lambda(1-2\alpha)+\mu(1-2\beta)\}\zeta(\zeta-\lambda-\mu)+\lambda\mu(\lambda+\mu)(1-\alpha-\beta),\\ C &= -\{(\zeta-\mu)-(\lambda\alpha-\mu\beta)\}\{(\zeta-\lambda)+(\lambda\alpha-\mu\beta)\}.\end{aligned} \tag{2.15}$$

Here put $D = B^2-4AC$, it follows by direct calculation that

$$D = (2\zeta-\lambda-\mu)^2(\zeta-\gamma_1)(\zeta-\gamma_2)(\zeta-\gamma_3)(\zeta-\gamma_4), \tag{2.16}$$

where γ_i's are given by

$$\begin{aligned}\gamma_1 &= \frac{1}{2}\left((\lambda+\mu)-\sqrt{(\lambda+\mu)^2-4\lambda\mu\left(\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta}\right)^2}\right),\\ \gamma_2 &= \frac{1}{2}\left((\lambda+\mu)-\sqrt{(\lambda+\mu)^2-4\lambda\mu\left(\sqrt{(1-\alpha)(1-\beta)}+\sqrt{\alpha\beta}\right)^2}\right),\\ \gamma_3 &= \frac{1}{2}\left((\lambda+\mu)+\sqrt{(\lambda+\mu)^2-4\lambda\mu\left(\sqrt{(1-\alpha)(1-\beta)}+\sqrt{\alpha\beta}\right)^2}\right),\\ \gamma_4 &= \frac{1}{2}\left((\lambda+\mu)+\sqrt{(\lambda+\mu)^2-4\lambda\mu\left(\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta}\right)^2}\right).\end{aligned} \tag{2.17}$$

Swap p and q, and replace p by $1-p$ or q by $1-q$ if necessary, we may assume that $\lambda \geq \mu$ and $\alpha \leq \beta \leq \frac{1}{2}$ without any loss of generality. Now we shall pay our attention upon the case where strictly $\lambda > \mu$. It is easy to have the inequalities

$$0 \leq \gamma_1 \leq \gamma_2 \leq \mu < \lambda \leq \gamma_3 \leq \gamma_4 \leq \lambda+\mu. \tag{2.18}$$

In this case, $G_\ell(\zeta)$, can be written in the form

$$G_\ell(\zeta) = \frac{1}{2\zeta(\zeta-\lambda)(\zeta-\mu)(\zeta-\lambda-\mu)}\times$$
$$\Big(-\{\lambda(1-2\alpha)+\mu(1-2\beta)\}\zeta(\zeta-\lambda-\mu)-\lambda\mu(\lambda+\mu)(1-\alpha-\beta)$$
$$+\sqrt{(2\zeta-\lambda-\mu)^2(\zeta-\gamma_1)(\zeta-\gamma_2)(\zeta-\gamma_3)(\zeta-\gamma_4)}\Big), \tag{2.19}$$

where the branch of the analytic square root should be determined by the condition (2.9) as we mentioned before. We shall find the probability measure ν by applying the Stieltjes inversion formula on $G_\ell(\zeta)$.

We consider the case that $\alpha < \beta$. It is the most generic case where $G_\ell(\zeta)$ has two removable singularities and two simple poles. Taking care of the choices of the branch of the analytic square root in $G_\ell(\zeta)$, we obtain the residues at simple poles 0 and λ as

$$\mathrm{Res}(0) = 1-\alpha-\beta, \quad \mathrm{Res}(\lambda) = \beta-\alpha. \tag{2.20}$$

Here $z=\mu$ and $z=\lambda+\mu$ are removable singularities. As we have done before, from Stieltjes inversion formula, it follows that ν is absolutely continuous with respect to Lebesgue measure on the intervals $[\gamma_1,\gamma_2]$ and $[\gamma_3,\gamma_4]$. For $t\in[\gamma_1,\gamma_2]$, it is easy to see that the density is given by

$$\begin{aligned} f_1(t) &= -\frac{1}{\pi}\lim_{\varepsilon\to+0}\mathrm{Im}\ G_\ell(t+i\varepsilon) \\ &= \frac{\left(t-\frac{\lambda+\mu}{2}\right)\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)} \end{aligned} \tag{2.21}$$

and, for $t\in[\gamma_3,\gamma_4]$,

$$\begin{aligned} f_2(t) &= -\frac{1}{\pi}\lim_{\varepsilon\to+0}\mathrm{Im}\ G_\ell(t+i\varepsilon) \\ &= \frac{-\left(t-\frac{\lambda+\mu}{2}\right)\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)}. \end{aligned} \tag{2.22}$$

Thus, we have the probability measure as

$$d\nu = f_1(t)\chi_{[\gamma_1,\gamma_2]}dt + f_2(t)\chi_{[\gamma_3,\gamma_4]}dt + (1-\alpha-\beta)\delta_0 + (\beta-\alpha)\delta_\lambda. \tag{2.23}$$

This measure has the two intervals and the two points as its support.

In the other cases, we can also find the probability measure without much difficulties by using Stieltjes inversion formula via the similar arguments. So we would like to omit the details.

Theorem 2.2. *Let $\{p,q\}$ be a free pair of projections with $\phi(p)=\alpha$ and $\phi(q)=\beta$, and let λ and μ are positive scalars. Put $\ell=\lambda p+\mu q$ then the distribution ν for the element ℓ is given in the following:*

(I) $\lambda>\mu$

(i) $\alpha<\beta$,

$$\begin{aligned} d\nu = &\frac{-\left|t-\frac{\lambda+\mu}{2}\right|\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi t(t-\lambda)(t-\mu)(t-\lambda-\mu)}\chi_{[\gamma_1,\gamma_2]\cup[\gamma_3,\gamma_4]}dt \\ &+(1-\alpha-\beta)\delta_0+(\beta-\alpha)\delta_\lambda \end{aligned} \tag{2.24}$$

(ii) $\alpha=\beta\neq\frac{1}{2}$.

$$\begin{aligned} d\nu = &\frac{-\left|t-\frac{\lambda+\mu}{2}\right|}{\pi t(t-\lambda-\mu)}\sqrt{-\frac{(t-\gamma_1)(t-\gamma_4)}{(t-\lambda)(t-\mu)}}\chi_{[\gamma_1,\lambda]\cup[\mu,\gamma_4]}dt \\ &+(1-2\alpha)\delta_0 \end{aligned} \tag{2.25}$$

(iii) $\alpha=\beta=\frac{1}{2}$,

$$d\nu = \frac{\left|t-\frac{\lambda+\mu}{2}\right|}{\pi\sqrt{-t(t-\lambda)(t-\mu)(t-\lambda-\mu)}}\chi_{[0,\lambda]\cup[\mu,\lambda+\mu]}dt \tag{2.26}$$

(II) $\lambda=\mu$;

(i) $\alpha<\beta$,

$$\begin{aligned} d\nu = &\frac{\sqrt{-(t-\gamma_1)(t-\gamma_2)(t-\gamma_3)(t-\gamma_4)}}{\pi|t(t-\lambda)(t-2\lambda)|}\chi_{[\gamma_1,\gamma_2]\cup[\gamma_3,\gamma_4]}dt \\ &+(1-\alpha-\beta)\delta_0+(\beta-\alpha)\delta_\lambda \end{aligned} \tag{2.27}$$

(ii) $\alpha=\beta\neq\frac{1}{2}$

$$d\nu = \frac{\sqrt{-(t-\gamma_1)(t-\gamma_4)}}{-\pi t(t-2\lambda)}\chi_{[\gamma_1,\gamma_4]}dt+(1-2\alpha)\delta_0 \tag{2.28}$$

(iii) $\alpha=\beta=\frac{1}{2}$.

$$d\nu = \frac{1}{\pi\sqrt{-t(t-2\lambda)}}\chi_{[0,2\lambda]}dt. \tag{2.29}$$

where γ_i's are given by (2.17).

Of course, the last two cases in Theorem 2.2 are included in the case of $n=2$ of Theorem 2.1 and the last one is nothing but the arcsin law on the interval $[0,2\lambda]$.

In the rest of this section, we should like to make some comments on the measures which have been obtained in this section, as applications. The special cases of these measures have been obtained as the spectral measures of the adjacency operators of some infinite graphs and the Plancherel measures for some infinite discrete groups. Here we shall show their definitions and explain how they connect to our results.

Definition 2.3. Let $\mathcal{G} = (V, E)$ be an unoriented infinite graphs with the set of vertices V and one of edges E. One consider the Hilbert space $\ell^2(V)$ of all the square summable functions on V. Suppose $\mathcal{G}$ is *uniformly locally finite*, that is, $\deg(\mathcal{G}) = \sup\{\deg(u) \ : \ u \in V\} < \infty$, where $\deg(u)$ is the number of edges emanating from u.

Then the bounded self-adjoint operator A on $\ell^2(V)$ called the *adjacency operator* of $\mathcal{G}$, is definied by

$$(Af)(u) = \sum_{(u,v)} f(v) \qquad f \in \ell^2(V), \tag{2.30}$$

where (u, v) forms an edge.

Many references of the papers concerning with the adjacency operators can be found in [MW], which contains the good survey on spectra of many interested infinite graphs.

On the measures in Theorem 2.1, the some of them have been obtained as the spectral measures of the adjacency operators of the infinite distance-regular graphs.

Definition 2.4. A connected graph $\mathcal{G}$ is called *distance-regular* if there exists a function $f : (\mathbb{N}_0)^3 \longrightarrow \mathbb{N}_0$ such that for all $u, v \in V(\mathcal{G})$ and $j, k \in \mathbb{N}_0$,

$$^{\#}\{w \in V(\mathcal{G}) : d(u,w) = j, d(v,w) = k\} = f(j, k, d(u,v)), \tag{2.31}$$

where $V(\mathcal{G})$ is the set of all vatices of the graph $\mathcal{G}$ and, as usual, $d(u, v)$ is the distance between u and v, the length of a shortest walk from u to v.

The infinite distance-regular graphs have been completely characterized [Iv]. They are tree-like graphs and parameterized by two integers $m, s \geq 2$. The infinite distance-regular graph $D_{m,s}$ can be obtained from the biregular tree $T_{m,s}$. Here, the biregular $T_{m,s}$ is an infinite tree where the vertex degree is constant on each of the two bipartite classes, with values m and s, respectively. The set of vertices of the infinite distance-regular graph $D_{m,s}$ is the bipartite block of degree m, and two vertices constitute an edge if and only if their distance in $T_{m,s}$ is two. Hence, each vertex of $D_{m,s}$ lies in the intersection of exactly m copies of the finite complete graph K_s, in particular, $D_{m,2}$ is nothing but the m-homogeneous tree T_m, and the spectral theory of the graph $D_{m,s}$ is similar to that of the homogenuous tree.

We consider the free product group

$$G = \underbrace{\mathbb{Z}_s * \mathbb{Z}_s * \cdots * \mathbb{Z}_s}_{m} \tag{2.32}$$

and the reduced group C^*-algebra $C_r^*(G)$. Let u_i $(i = 1, 2, ..., m)$ be the unitary generator of each cyclic group in $C_r^*(G)$. Then it is easy to see that, for all i,

$$p_i = \frac{1}{s}\sum_{j=1}^{s}(u_i)^j \tag{2.33}$$

is a projection with $\tau_G(p_i) = 1/s$. Furthermore, $(p_i)_{i=1,2,\dots,m}$ is a free family of projections in a C^*-probability space $(C^*(G), \tau_G)$; see Example 1.3.

From the definitions of the free product and of the infinite distance-regular graph, it is clear that there exsists a bijection between the set of vertices of the graph $D_{m,s}$ and the group G, Then the adjacency operator A can be represented as

$$A = \sum_{i=1}^{m}\left(u_i + (u_i)^2 + \cdots + (u_i)^{s-1}\right) = \sum_{i=1}^{m}(sp_i - 1) = s\sum_{i=1}^{s} p_i - m \cdot 1 \tag{2.34}$$

in $C_r^*(G)$. Now Theorem 2.1 is applicatable with $n = m$, $\lambda = s$, and $\alpha = 1/s$. Making m-shift, we have the spectral measure $\nu_{m,s}$ for the adjacency operator of $D_{m,s}$ in the following :

Writing

$$I_{m,s} = [s-2-2\sqrt{(m-1)(s-1)}, s-2+2\sqrt{(m-1)(s-1)}] \tag{2.35}$$

and

$$f_{m,s} = \frac{-m\sqrt{-(t-s+2)^2+4(m-1)(s-1)}}{2\pi(t+m)(t-m(s-1)}, \tag{2.36}$$

we obatain

$$d\nu_{m,s} = \begin{cases} f_{m,s}\chi_{I_{m,s}}dt & \text{if } m \geq s, \\ f_{m,s}\chi_{I_{m,s}}dt + \left(1-\frac{m}{s}\right)\delta_{-m} & \text{if } m < s. \end{cases} \tag{2.37}$$

Remark 2.5. The measures that we obtained in Theorem 2.1 can be also found in [Co] and [CT]. Especially in [CT], they calculated the measure for which a sequence of polynomials generated from a recursion formula with constant Jacobi parameters, is orthogonal.

Next let us make a comment on the measures in Theorem 2.2. In [CS1], they consider the free product group $G = \mathbb{Z}_r * \mathbb{Z}_s$, where $r > s \geq 2$ and the length for the elements of G is defined. They study the convolution C^*-algebra generated by the characteristic function χ_1 on the elements of the length 1 and obtain the associated Plancherel measure. This measure can be regarded as the special case of ours as follows :

Let u_1 and u_2 be the unitary generators of the cyclic groups in the reduced C^*-algebra $C_r^*(G)$ for $\mathbb{Z}_r$ and $\mathbb{Z}_s$, respectively. Then their convolution operator T_{χ_1} associated to the characteristic function χ_1 is in the form

$$T_{\chi_1} = \sum_{i=1}^{r-1}(u_1)^i + \sum_{j=1}^{s-1}(u_2)^j. \tag{2.38}$$

As we mentioned before, $\sum_{i=1}^{r-1}(u_1)^i$ can be written as $rp_1 - 1$, where p_1 is a projection of trace $1/r$. Similarly, we have $\sum_{j=1}^{s-1}(u_2)^j = sp_2 - 1$ with a projection p_2 of trace $1/s$. Hence we can write

$$T_{\chi_1} = rp_1 + sp_2 - 2 \tag{2.39}$$

and p_1 and p_2 are free. Now it is clear that the Plancherel measure can be obtained as the special case of Theorem 2.2; see also [CS2].

As we stated in the beginning of this section, if the family $\{(\alpha_i, \lambda_i)\}_{i=1,2,\ldots,n}$ is constituted from at most two diffrent pairs then we can find the generating function $G(\zeta)$ exactly. That is, in the case where

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha, \quad \alpha_{m+1} = \cdots = \alpha_n = \beta,$$
$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda, \quad \lambda_{m+1} = \cdots = \lambda_n = \mu.$$

Thus, for example, we can also obtain the Plancherel measure for the group of the free product of k copies of $\mathbb{Z}_r$ and m copies of $\mathbb{Z}_s$.

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